

Smoluchowski problem for degenerate Bose gases

Anatoly V. Latyshev and Alexander A. Yushmanov

*Department of Mathematical Analysis and Department of Theoretical Physics,
Moscow State Regional University, 105005, Moscow, Radio st., 10-A*

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We construct a kinetic equation simulating the behavior of degenerate quantum Bose gases with the collision rate proportional to the molecule velocity. We obtain an analytic solution of the half-space boundary-value Smoluchowski problem of the temperature jump at the interface between the degenerate Bose gas and the condensed phase.

Keywords: degenerate quantum Bose gas, Bose — Einstein condensate, collision integral, temperature jump, Kapitza resistance.

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1. INTRODUCTION

In recent years, the behavior of quantum gases has increased interest. In particular, this is related to the development of experimental procedure for producing and studying quantum gases at extremely low temperatures [1]. In the majority of papers, bulk properties of quantum gases have been studied [2] and [3]. At the same time, it is obvious that it is important to take boundary effects on the properties of such systems into account. In particular, such a phenomenon as the temperature jump at the interface between a gas and a condensed (in particular, solid) body in the presence of a heat flux normal to the surface is important. Such a temperature jump is frequently called the Kapitza temperature jump [4]. We also mention a paper where the problem of boundary conditions for the motion of superliquids was considered [5].

The problem of a temperature jump in a quantum Fermi gas was studied in [6], where an analytic solution for an arbitrary degree of gas degeneracy was obtained. A similar problem for a Bose gas was considered in [7], where the gas was assumed to be nondegenerate, i.e., it was assumed that there was no Bose — Einstein condensate.

The present paper is devoted to solving the problem of a temperature jump in a degenerate Bose gas analytically. The presence of a Bose — Einstein condensate [8] leads to a considerable modification of both the problem statement and its solution method. In this case, a kinetic equation with a model collision integral is used to describe kinetic processes. We assume that the boundary

conditions at the surface are purely diffusive.

2. KINETIC EQUATION

To describe the gas behavior, we use a kinetic equation with a model collision integral analogous to that used to describe a classical gas. In this case, we take into account the quantum character of the Bose gas and the presence of the Bose — Einstein condensate.

For a rarefied Bose gas, the evolution of the molecule distribution function f can be described by the kinetic equation

$$\frac{\partial f}{\partial t} + \frac{\partial \mathcal{E}}{\partial \mathbf{p}} \nabla f = I[f],$$

where \mathcal{E} is the kinetic energy of molecules, \mathbf{p} is the molecule momentum, and $I[f]$ is the collision integral.

In the case of the kinetic description of a degenerate Bose gas, we must take into account that the properties of the Bose — Einstein condensate can change as functions of the space and time coordinates. In other words, we must consider a two-liquid model (more precisely, a two-fluid model, because we consider a gas rather than a liquid). We let $\rho_c = \rho_c(\mathbf{r}, t)$ and $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$ denote the respective density and velocity of the Bose condensate — Einstein. We then have the expressions [9]

$$\mathbf{j} = \rho_c \mathbf{u}, \quad \mathbf{Q} = \frac{\rho_c u^2}{2} \mathbf{u},$$

$$\Pi_{ik} = \rho_c u_i u_k.$$

for the densities \mathbf{j} and \mathbf{Q} of the respective mass and energy fluxes and for the momentum flux tensor Π_{ik} of

the Bose condensate — Einstein (under the assumption that the chemical potential is zero).

The conservation laws for the number of particles, energy, and momentum require that the relations

$$\begin{aligned}\frac{\partial \rho_c}{\partial t} + \nabla \mathbf{j} &= - \int I[f] d\Omega_B, \\ \frac{\partial E_c}{\partial t} + \nabla \mathbf{Q} &= - \int \frac{p^2}{2m} I[f] d\Omega_B, \\ \frac{\partial (\rho_c \mathbf{u})}{\partial t} + \nabla \Pi &= - \int \mathbf{p} I[f] d\Omega_B\end{aligned}$$

be satisfied, where s is the molecule spin,

$$d\Omega_B = \frac{(2s+1)d^3p}{(2\pi\hbar)^3}, \quad E_c = \frac{\rho_c u^2}{2},$$

where \hbar is the Planck constant.

In what follows, we are interested in the case of stationary motion with small velocities (compared with the thermal velocities). We note that for the Bose condensate, the quantities \mathbf{Q} and Π_{ik} , are depend nonlinearly on the velocity (they are proportional to the respective third and second powers of the velocity). Therefore, in the approximation linear in the velocity \mathbf{u} , the energy and momentum conservation laws can be written as

$$\int \frac{p^2}{2m} I[f] d\Omega_B$$

and

$$\int \mathbf{p} I[f] d\Omega_B = 0.$$

According to the Bogolyubov theory, the relation for the excitation energy $\varepsilon(p)$ [8]

$$\varepsilon(p) = \left[u^2 p^2 + \left(\frac{p^2}{2m} \right)^2 \right]^{1/2}, \quad (1)$$

where

$$u = \left(\frac{4\pi\hbar^2 a n}{m^2} \right)^{1/2}.$$

Here a is the scattering length for gas molecules, n is the concentration, m is the mass, and \mathbf{p} is the momentum of the gas molecule, holds for a weakly interacting Bose gas. The parameter a characterizes the interaction force of gas molecules and can be assumed to be small for a weakly interacting gas.

The relation $u^2 \ll kT/m$, where k is the Boltzmann constant and T is the gas temperature, holds for sufficiently small a . In this case, we can neglect the first term in the brackets in (1). The expression for the energy $\mathcal{E}(p)$ takes the same form as in the case of noninteracting molecules:

$$\mathcal{E}(p) = \frac{p^2}{2m}.$$

We now consider the widely used kinetic equation in the Boltzmann — Krook — Welander form with the molecule collision rate proportional to the molecule velocity [6, 7, 10]

$$\frac{\partial f}{\partial t} + (\mathbf{v} \nabla) f = \nu_0 w (f_M^* - f). \quad (2)$$

Here, f is the distribution function, \mathbf{v} is the molecule velocity, $w = |\mathbf{v} - \mathbf{v}_0|$, \mathbf{v}_0 is the mean-mass gas velocity, f_M^* is the Maxwell distribution function,

$$f_M^* = n_* \left(\frac{m}{2\pi k T_*} \right)^{3/2} \exp \left[-\frac{m}{2k T_*} (\mathbf{v} - \mathbf{u}_*)^2 \right],$$

and ν_0 is a model parameter corresponding to the inverse mean free path l of a molecule, $\nu_0 \sim 1/l$.

The parameters in the formula for f_M^* , namely, n_* , T_* and \mathbf{u}_* , can be determined from the conservation laws for the number of molecules, momentum, and energy

$$\int w f d\Omega_M = \int w f_M^* d\Omega_M, \quad (3a)$$

$$\int w \mathbf{v} f d\Omega_M = \int w \mathbf{v} f_M^* d\Omega_M, \quad (3b)$$

$$\int w \frac{m}{2} (\mathbf{v} - \mathbf{u})^2 f d\Omega_M = \int w \frac{m}{2} (\mathbf{v} - \mathbf{u})^2 f_M^* d\Omega_M, \quad (3c)$$

where

$$d\Omega_M = d^3v.$$

The conservation law for the number of particles in the normal state is inapplicable because the transition of particles to the Bose — Einstein condensate can occur. As mentioned above, the effect of the condensate on the energy and momentum conservation laws can be neglected in the approximation linear in \mathbf{u} .

We note that Eq. (2) corresponds to the assumption that the mean free path of molecules is constant (it is independent of molecule velocities). It hence follows that

Eq. (2) to a greater extent corresponds to the model in which the molecules are regarded as solid spheres.

We consider a generalization of Eq. (2) to the case of a degenerate Bose gas. We assume that the general structure of Eq. (2) is preserved, but by a function f_M^* , we must mean the Bose — Einstein distribution (Bosean) with a zero chemical potential [8]

$$f_B^* = \left[\exp \left(\frac{m}{2kT_*} (\mathbf{v} - \mathbf{u}_*)^2 \right) - 1 \right]^{-1}.$$

Here, the parameters T_* and \mathbf{u}_* , are determined by the second and third conditions in (3). In this case, we have

$$d\Omega_B = \frac{2s+1}{(2\pi\hbar)^3} d^3p$$

instead $d\Omega_M$.

We assume that the mass velocity of the gas is much less than the mean thermal velocity of the molecules and the typical temperature variations along the mean free path of molecules are small compared with the gas temperature. The problem can be linearized under these assumptions.

We seek the distribution function in the form

$$f = f_B^s(v) + \varphi(t, \mathbf{r}, \mathbf{v})g(v),$$

where

$$f_B^s(v) = \frac{1}{\exp(\beta_s v^2) - 1}, \quad \beta_s = \frac{m}{2kT_s},$$

φ is a new unknown function, T_s is the surface temperature, and

$$g(v) = -\frac{\partial}{\partial \varepsilon_s} f_B^s, \quad \varepsilon_s = \beta_s v^2.$$

We introduce the notation

$$\mathbf{C} = \sqrt{\beta_s} \mathbf{v}, \quad \varepsilon_* = \frac{m}{2kT_*} (\mathbf{v} - \mathbf{u}_*)^2.$$

Taking this notation into account, we have

$$f_B^*(\varepsilon_*) = \frac{1}{\exp(\varepsilon_*) - 1},$$

$$f_B^s(C) = \frac{1}{\exp(C^2) - 1},$$

$$g(C) = \frac{\exp(C^2)}{(\exp(C^2) - 1)^2}.$$

We linearize the local Bose — Einstein distribution f_B^* , passing to dimensionless quantities. We note that

$$\varepsilon_* = \frac{T_s}{T_*} [\beta_s (\mathbf{v} - \mathbf{u}_*)^2] = \frac{T_s}{T_*} [(\mathbf{C} - \mathbf{W})^2],$$

where

$$\mathbf{W} = \sqrt{\beta_s} \mathbf{u}_*.$$

Taking $T_* = T_s + \delta T_s$ into account, we obtain

$$\varepsilon_* = C^2 - \frac{\delta T_*}{T_s} C^2 - 2\mathbf{C}\mathbf{W},$$

whence we find

$$\delta \varepsilon_* = -2\mathbf{C}\mathbf{W} - C^2 \frac{\delta T_*}{T_s},$$

where

$$\delta \varepsilon_* = \varepsilon_* - \varepsilon_s, \quad \varepsilon_s = C^2.$$

Consequently,

$$f_B^* = f_B^s + \left(\frac{\partial f_B^*}{\partial \varepsilon_*} \right)_{\varepsilon_* = \varepsilon_s} \delta \varepsilon_*,$$

or

$$f_B^* = f_B^s + g(C) \left[2\mathbf{C}\mathbf{W} + C^2 \frac{\delta T_*}{T_s} \right].$$

We note that the quantity w in Eq. (2) can be replaced with v in the approximation under consideration. We introduce the dimensionless quantities $t^* = t\nu_0/\beta_s$ and $\mathbf{r}^* = \mathbf{r}\nu_0$ and omit the asterisks on these quantities below. It is now clear that Eq. (2) (in the dimensionless variables) becomes

$$\frac{\partial \varphi}{\partial t} + (\mathbf{C}\nabla)\varphi = 2C(\mathbf{C}\mathbf{W}) + C^3 \frac{\delta T_*}{T_s} - C\varphi. \quad (4)$$

The parameters of this equation can be found from the momentum and energy conservation laws (relations (3)), which now become

$$\int (L\varphi) C^2 g(C) d^3C = 0,$$

$$\int (L\varphi) \mathbf{C} g(C) d^3C = 0,$$

where

$$L\varphi = 2C\mathbf{C}\mathbf{W} + C^3 \frac{\delta T_*}{T_s} - C\varphi.$$

From this system, we find

$$\mathbf{W} = \frac{3}{8\pi g_0} \int \varphi \mathbf{C} g(C) C d^3 C,$$

$$\frac{\delta T_*}{T_s} = \frac{1}{4\pi g_2} \int \varphi g(C) C^3 d^3 C,$$

where

$$g_n = \int_0^\infty C^{5+n} g(C) dC, \quad n = 0, 1, 2,$$

In this case, we have

$$\begin{aligned} g_0 &= \int_0^\infty g(C) C^5 dC = \int_0^\infty \frac{\exp(C^2) C^5 dC}{(\exp(C^2) - 1)^2} = \\ &= -2 \int_0^\infty C \ln(1 - \exp(-C^2)) dC = \frac{\pi^2}{6} = 1.64493, \end{aligned}$$

$$g_1 = \int_0^\infty g(C) C^6 dC = 2.22912,$$

$$g_2 = \int_0^\infty g(C) C^7 dC = 3.60617.$$

We represent Eq. (4) in the standard form

$$\begin{aligned} \frac{\partial \varphi}{\partial t} + (\mathbf{C} \nabla) \varphi + C \varphi(t, \mathbf{r}, \mathbf{C}) = \\ = \frac{C}{4\pi} \int k(\mathbf{C}, \mathbf{C}') \varphi(t, \mathbf{r}, \mathbf{C}') C' g(C') d^3 C', \end{aligned} \quad (5)$$

где

$$k(\mathbf{C}, \mathbf{C}') = \frac{3}{g_0} \mathbf{C} \mathbf{C}' + \frac{1}{g_2} C^2 C'^2.$$

3. PROBLEM STATEMENT

A degenerate Bose gas occupies the half-space $x > 0$ above the plane surface in the problem under consideration. A heat flux Q normal to the surface is maintained in the gas. We let T_0 denote the gas temperature far from the surface. The quantity T_0 differs from the surface temperature T_s if there is a heat flux. We let $\Delta = T_0 - T_s$ be the difference between these

temperatures. The quantity Δ is called the temperature jump (the Kapitza temperature jump in the case of low temperatures).

The problem is to find ΔT as a function of the heat flux Q . Taking into account that the problem is linear, we can write

$$\Delta T = T_Q Q.$$

The quantity T_Q is called the coefficient of the temperature jump. Another notation can be used:

$$Q = R \Delta T,$$

where R is called the Kapitza resistance.

Taking into account that the problem is stationary and that the function φ is independent of the coordinates and z , we simplify Eq. (5):

$$\begin{aligned} \mu \frac{\partial \varphi}{\partial x} + \varphi(x, \mu, C) = \\ = \frac{1}{2} \int_{-1}^1 \int_0^\infty k(\mu, C; \mu', C') \varphi(x, \mu', C') d\Omega(C'), \end{aligned} \quad (6)$$

where

$$\mu = \frac{C_x}{C}, \quad d\Omega(C') = g(C') C'^3 d\mu' d\epsilon',$$

$$k(C, \mu; C', \mu') = \frac{3}{g_0} C \mu C' \mu' + \frac{1}{g_2} C^2 C'^2.$$

It is easy to verify that Eq. (6) has the particular solutions

$$\varphi_1 = \mu C, \quad \varphi_2 = C^2.$$

Consequently, the function

$$\varphi_{as}(x, \mu, C) = B \mu C + \varepsilon_T C^2, \quad (7)$$

where B is proportional to the heat flux, is an asymptotic distribution function (as $x \rightarrow +\infty$).

Assuming that the reflection of the molecules from the wall is purely diffusive, we now formulate the boundary conditions

$$\varphi(0, \mu, C) = 0, \quad 0 < \mu < 1, \quad (8)$$

$$\varphi(x, \mu, C) =$$

$$= \varphi_{as}(x, \mu, C) + o(1), \quad x \rightarrow +\infty, \quad -1 < \mu < 0. \quad (9)$$

4. REDUCTION TO THE ONE-VELOCITY PROBLEM AND THE SEPARATION OF VARIABLES

We seek the solution of problem (6)–(9) in the form

$$\varphi(x, \mu, C) = Ch_1(x, \mu) + C^2h_2(x, \mu). \quad (10)$$

We obtain the system of equations

$$\begin{aligned} \mu \frac{\partial h_1}{\partial x} + h_1(x, \mu) &= \\ &= \frac{3}{2}\mu \int_{-1}^1 \mu' h_1(x, \mu') d\mu' + \frac{3g_1}{2g_0}\mu \int_{-1}^1 \mu' h_2(x, \mu') d\mu', \\ \mu \frac{\partial h_2}{\partial x} + h_2(x, \mu) &= \\ &= \frac{g_1}{2g_2} \int_{-1}^1 h_1(x, \mu') d\mu' + \int_{-1}^1 h_2(x, \mu') d\mu'. \end{aligned}$$

We represent this system of equations with respect to the column vector

$$h(x, \mu) = \begin{bmatrix} h_1(x, \mu) \\ h_2(x, \mu) \end{bmatrix}$$

in the vector form

$$\mu \frac{\partial h}{\partial x} + h(x, \mu) = \frac{1}{2} \int_{-1}^1 K(\mu, \mu') h(x, \mu') d\mu', \quad (11)$$

where $K(\mu, \mu')$ is the kernel of Eq. (11),

$$K(\mu, \mu') = \begin{bmatrix} 3\mu\mu' & 3\frac{g_1}{g_0}\mu\mu' \\ \frac{g_1}{g_2} & 1 \end{bmatrix}.$$

Using (10), we transform boundary conditions (8) and (9) into

$$h(0, \mu) = \mathbf{0}, \quad 0 < \mu < 1, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (12)$$

$$h(x, \mu) =$$

$$= h_{as}(x, \mu) + o(1), \quad x \rightarrow +\infty, \quad -1 < \mu < 0, \quad (13)$$

where $h_{as}(x, \mu)$ is the asymptotic distribution function

$$h_{as}(x, \mu) = \begin{bmatrix} B\mu \\ \varepsilon_T \end{bmatrix}.$$

We note that the equation kernel can be represented in the form

$$K(\mu, \mu') = K_0 + 3\mu\mu'K_1,$$

$$K_0 = \begin{bmatrix} 0 & 0 \\ \frac{g_1}{g_2} & 1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 1 & \frac{g_1}{g_0} \\ 0 & 0 \end{bmatrix}.$$

The general method for the separation of variables yields the expression

$$h_\eta(x, \mu) = \exp\left(-\frac{x}{\eta}\right) \Phi(\eta, \mu). \quad (14)$$

Substituting (14) in (11), we obtain the characteristic equation

$$(\eta - \mu) \Phi(\eta, \mu) = \frac{\eta}{2} K_0 n^{(0)}(\eta) + \frac{3}{2} \mu \eta K_1 n^{(1)}(\eta),$$

where

$$n^{(k)}(\eta) = \int_{-1}^1 \mu^k \Phi(\eta, \mu) d\mu, \quad k = 0, 1.$$

It is obvious, that

$$n^{(1)}(\eta) = \eta(E - K_0)n^{(0)}(\eta),$$

where E is the unit matrix of the second order.

We obtain now the characteristic equation

$$(\eta - \mu) \Phi(\eta, \mu) = \frac{1}{2} \eta D(\mu\eta) n(\eta), \quad (15)$$

$$n(\eta) \equiv n^{(0)}(\eta) = \int_{-1}^1 \Phi(\eta, \mu) d\mu,$$

where

$$D(\mu\eta) = K_0 + 3\mu\eta K_1(E - K_0) =$$

$$= \begin{bmatrix} 3g\mu\eta & 0 \\ \frac{g_1}{g_2} & 1 \end{bmatrix},$$

$$g = 1 - \frac{g_1^2}{g_0g_2} = 0.96288.$$

For $\eta \in (-1, 1)$, we use Eqs. (15) in the class of generalized functions [11] to find the eigenvectors

$$\Phi(\eta, \mu) = F(\eta, \mu)n(\eta)$$

of the continuous spectrum, where

$$F(\eta, \mu) = \frac{1}{2}\eta D(\mu\eta)P\frac{1}{\eta - \mu} + \Lambda(\eta)\delta(\eta - \mu),$$

$$n(\eta) = \begin{bmatrix} n_1(\eta) \\ n_2(\eta) \end{bmatrix} = \int_{-1}^1 \Phi(\eta, \mu) d\mu, \quad (16)$$

$F(\eta, \mu)$ is the eigen matrix-function, the symbol Px^{-1} denotes the principal value of the integral in the integration of x^{-1} , $\delta(x)$ is the delta function, and $\Lambda(z)$ is the dispersion matrix,

$$\Lambda(z) = E + \frac{z}{2} \int_{-1}^1 \frac{D(\mu z)}{\mu - z} d\mu.$$

The dispersion matrix has following elements

$$\Lambda_{11}(z) = 1 + 3gz^2\lambda_0(z) \equiv \lambda_1(z),$$

$$\Lambda_{12}(z) \equiv 0,$$

$$\Lambda_{21}(z) = \frac{g_1}{g_2}\lambda_0(z) - \frac{g_1}{g_2},$$

$$\Lambda_{22}(z) = \lambda_0(z).$$

We represent the dispersion matrix in the explicit form

$$\Lambda(z) = \lambda_0(z)D(z^2) + D_0,$$

where

$$D_0 = \begin{bmatrix} 1 & 0 \\ -\frac{g_1}{g_2} & 0 \end{bmatrix},$$

$$\lambda_0(z) = 1 + \frac{z}{2} \int_{-1}^1 \frac{du}{u - z}.$$

Its determinant, called the dispersion function, is given by

$$\lambda(z) = \det \Lambda(z) = \lambda_0(z)\lambda_1(z).$$

5. EIGENVECTORS OF THE DISCRETE SPECTRUM

By definition [7], the set of zeros of the dispersion function is called the discrete spectrum.

The function $\lambda_0(z)$ has a single double zero at the point at infinity $z_i = \infty$. Two eigensolutions of Eq. (11) correspond to this zero (they coincide with the eigenvectors of the characteristic equation):

$$h_0(x, \mu) = \mu \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad h_1(x, \mu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We find zeros of $\lambda_1(z)$ outside the cut $[-1, 1]$. It is obvious that $\lambda_1(\infty) = 1 - g = 0.03712 > 0$. We take the contour γ_ε (see Fig. 1) that encompasses the cut $[-1, 1]$ at a distance ε from it such that there are no zeros of the function $\lambda_1(z)$ inside the contour. The number N of zeros of $\lambda_1(z)$ outside the contour is equal to the increment of the argument of this function according to the argument principle [12], i.e.,

$$N = (2\pi)^{-1} \Delta_{\gamma_\varepsilon} \arg \lambda_1(z),$$

where the symbol $\Delta_{\gamma_\varepsilon} f$ means the increment of f along γ_ε . Passing to the limit as $\varepsilon \rightarrow 0$ in this equality, we obtain

$$N = \frac{1}{2\pi} \Delta_{(-1,1)} \arg \frac{\lambda_1^+(\mu)}{\lambda_1^-(\mu)},$$

where $\lambda_1^\pm(\mu)$ are the boundary values of $\lambda_1(z)$ on the interval $(-1, 1)$ from above and from below,

$$\lambda_1^\pm(\mu) = \lambda_1(\mu) \pm 3i\pi g\mu^3.$$

Taking into account that $\operatorname{Re} \omega^+(\mu) = \omega(\mu)$ is an even function and $\operatorname{Im} \omega^+(\mu)$ is odd, we have

$$N = \frac{1}{\pi} \Delta_{(0,1)} \arg G(\mu),$$

$$G(\mu) = \frac{\lambda_1^+(\mu)}{\lambda_1^-(\mu)}.$$

Let $\theta_1(\mu) = \arg \lambda_1^+(\mu)$ be the principle value of the argument specified by the condition $\theta_1(0) = 0$. Because

$$\overline{\lambda_1^+(\mu)} = \lambda_1^-(\mu), \quad |\lambda_1^+(\mu)| = |\lambda_1^-(\mu)|,$$

we have $\arg G(\mu) = 2\theta_1(\mu)$ and therefore

$$N = \frac{2}{\pi} \Delta_{(0,1)} \theta_1(\mu).$$

has a nonzero solution because

$$\det \Lambda(\eta_0) \equiv \lambda(\eta_0) = 0.$$

We represent Eq. (17) in the form of two scalar equations

$$\lambda_1(\eta_0)n_1(\eta_0) = 0, \quad (18)$$

$$\frac{g_1}{g_2}[\lambda_0(\eta_0) - 1]n_1(\eta_0) + \lambda_0(\eta_0)n_2(\eta_0) = 0. \quad (19)$$

In view of the condition $\lambda_1(\eta_0) = 0$, it follows from Eq. (18) that the upper element $n_1(\eta_0)$ is arbitrary and nonzero. From Eq. (19), we now find the lower element of the vector $n(\eta_0)$,

$$n_2(\eta_0) = -\frac{g_1}{g_2} \left(1 - \frac{1}{\lambda_0(\eta_0)}\right) n_1(\eta_0).$$

From the equation

$$\lambda_1(\eta_0) = 1 + 3g\eta_0^2\lambda_0(\eta_0) = 0$$

we find

$$\lambda_0(\eta_0) = -\frac{1}{3g\eta_0^2}.$$

Hence, the vector $n(\eta_0)$ has the form

$$n(\eta_0) = \begin{bmatrix} 1 \\ -\frac{g_1}{g_2}(1 + 3g\eta_0^2) \end{bmatrix} n_1(\eta_0).$$

We substitute this vector in the second condition in (17) and obtain

$$\Phi(\eta_0) = \frac{1}{\eta_0 - \mu} \begin{bmatrix} 1 \\ -\frac{g_1}{g_2}\eta_0 \end{bmatrix},$$

for

$$n_1(\eta_0) = \frac{2}{3g\eta_0^2} = -2\lambda_0(\eta_0).$$

6. DECOMPOSITION OF THE SOLUTION INTO EIGENVECTORS: THE RIEMANN — HILBERT BOUNDARY VALUE PROBLEM

We show that the solution of boundary value problem (11)–(13) can be represented in the form of a decomposition, namely,

$$h(x, \mu) = h_{as}(x, \mu) + A_0 \exp\left(-\frac{x}{\eta_0}\right) \Phi(\eta_0, \mu) +$$

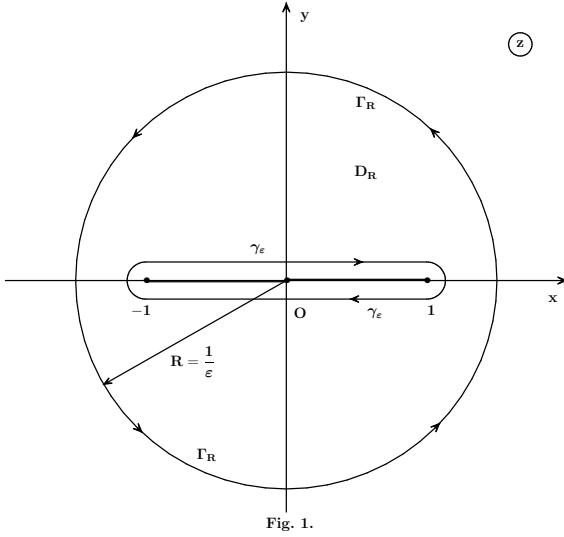


Fig. 1.

Fig. 1. Dependence of the temperature jump coefficient on the parameter 7: the specular reflection coefficient is $q = 0.3$ for curve 1, $q = 0.5$ for curve 2, and $q = 0.8$ for curve 3.

It is easy to see that the angle $\theta_1(\mu)$ on the cut $[0, 1]$ has an increment equal to π ; consequently, we have $N = 2$ (the number of zeros is equal to two). We let $\pm\eta_0$ ($\eta_0 = 1.27573$) denote these zeros. In view of the equality $\lambda_1(z) = \lambda_1(\bar{z})$, these zeros are real. Two eigensolutions $h_{\pm\eta_0}(x, \mu)$, where

$$h_{\eta_0}(x, \mu) = \Phi(\eta_0, \mu) \exp\left(-\frac{x}{\eta_0}\right), \quad (17a)$$

$$\Phi(\eta_0, \mu) = \frac{\eta_0}{2} \frac{D(\mu\eta_0)}{\eta_0 - \mu} n(\eta_0), \quad (17b)$$

correspond to them.

We note that the homogeneous equation

$$\Lambda(\eta_0)n(\eta_0) = 0$$

$$+ \int_0^\infty \exp(-\frac{x}{\eta}) F(\eta, \mu) A(\eta) d\eta, \quad (20)$$

where the unknowns are the constants ε_T and A_0 and a vector function $A(\eta)$ with the elements $A_j(\eta), j = 1, 2, 3$. Decomposition (20) can be represented in the form

$$\begin{aligned} h(x, \mu) &= h_{as}(x, \mu) + A_0 \exp(-\frac{x}{\eta_0}) \Phi(\eta_0, \mu) + \\ &+ \exp(-\frac{x}{\mu}) \Lambda(\mu) A(\mu) \theta_+(\mu) + \\ &+ \frac{1}{2} \int_0^1 \exp(-\frac{x}{\eta}) \eta D(\mu\eta) A(\eta) \frac{d\eta}{\eta - \mu}, \end{aligned} \quad (21)$$

where

$$\theta_+(\mu) = 1, \mu \in (0, 1); \quad \theta_+(\mu) = 0, \mu \notin (-1, 0).$$

Substituting decomposition (21) in boundary condition (12), we obtain a singular integral equation with the Cauchy kernel [12]

$$\begin{aligned} h_{as}(0, \mu) + A_0 \Phi(\eta_0, \mu) + \Lambda(\mu) A(\mu) + \\ + \frac{1}{2} \int_0^1 \eta D(\mu\eta) A(\eta) \frac{d\eta}{\eta - \mu} = \mathbf{0}, \quad 0 < \mu < 1. \end{aligned} \quad (22)$$

We introduce an auxiliary vector function

$$N(z) = \frac{1}{2} \int_0^1 \eta D(z\eta) A(\eta) \frac{d\eta}{\eta - z} \quad (23)$$

and the matrix

$$P(z) = \Lambda(z) D^{-1}(z^2).$$

Using the boundary values $N(z)$, $\Lambda(z)$ and $P(z)$ and the corresponding Sokhotski formulas, we reduce Eq. (22) to an inhomogeneous vector Riemann — Hilbert boundary value problem

$$\begin{aligned} P^+(\mu) [N^+(\mu) + h_{as}(0, \mu) + A_0 \Phi(\eta_0, \mu)] = \\ = P^-(\mu) [N^-(\mu) + h_{as}(0, \mu) + A_0 \Phi(\eta_0, \mu)], \quad 0 < \mu < 1. \end{aligned} \quad (24)$$

Let's lead reduction to the diagonal form of the problem (25). For this purpose we will lead to a diagonal kind the matrix

$$P(z) = \begin{bmatrix} \frac{\lambda_1(z)}{3gz^2} & 0 \\ -\frac{g_1}{3gg_2z^2} & \lambda_0(z) \end{bmatrix}.$$

The matrix bringing the matrix $P(z)$ to a diagonal kind, is that

$$S = \begin{bmatrix} 0 & -1 \\ 1 & \frac{g_1}{g_2} \end{bmatrix}, \quad \det S = 1, \quad S^{-1} = \begin{bmatrix} \frac{g_1}{g_2} & 1 \\ -1 & 0 \end{bmatrix}.$$

It is obvious that

$$S^{-1} P(z) S \equiv \Omega(z) = \text{diag} \left\{ \frac{\lambda_1(z)}{3gz^2}, \lambda_0(z) \right\}.$$

We first solve the homogeneous boundary value problem

$$P^+(\mu) X^+(\mu) = P^-(\mu) X^-(\mu), \quad 0 < \mu < 1, \quad (25)$$

corresponding to (24).

Clearly, that it is necessary to search for matrix $X(z)$ in the form

$$X(z) = S \text{diag} \{ U_1(z), U_2(z) \} S^{-1}.$$

The method for solving such problems was developed in [8]; therefore, we give the solution of problem (25) without a derivation,

$$X(z) = \begin{bmatrix} U_1(z) & 0 \\ \frac{g_1}{g_2} [U_0(z) - U_1(z)] & U_0(z) \end{bmatrix},$$

where

$$U_0(z) = z \exp(-V_0(z)),$$

$$U_1(z) = z \exp(-V_1(z)),$$

$$V_0(z) = \frac{1}{\pi} \int_0^1 \frac{\zeta_0(u) du}{u - z},$$

$$V_1(z) = \frac{1}{\pi} \int_0^1 \frac{\zeta_1(u) du}{u - z},$$

$$\zeta_0(u) = \theta_0(u) - \pi, \quad \zeta_0(0) = -\pi, \quad \zeta_0(1 - 0) = 0,$$

$$\zeta_1(u) = \theta_1(u) - \pi, \quad \zeta_1(0) = -\pi, \quad \zeta_1(1-0) = 0.$$

In this case, the angles have the form

$$\zeta_0(u) = -\frac{\pi}{2} - \arctg \frac{2\lambda_0(u)}{\pi u},$$

$$\zeta_1(u) = -\frac{\pi}{2} - \arctg \frac{2\lambda_1(u)}{3g\pi u^3}.$$

We now return to the solution of inhomogeneous problem (24). Substituting the matrix $P^+(\mu)$ found using (25) in (24), we obtain the problem of determining an analytic vector function from its jump

$$\begin{aligned} & \left[X^+(\mu) \right]^{-1} \left[N^+(\mu) + h_{as}(0, \mu) + A_0 \Phi(\eta_0, \mu) \right] = \\ & = \left[X^-(\mu) \right]^{-1} \left[N^-(\mu) + h_{as}(0, \mu) + A_0 \Phi(\eta_0, \mu) \right]. \end{aligned} \quad (26)$$

We note that

$$X^{-1}(z) = \begin{bmatrix} \frac{1}{U_1(z)} & 0 \\ \frac{g_1}{g_2} \left(\frac{1}{U_0(z)} - \frac{1}{U_1(z)} \right) & \frac{1}{U_0(z)} \end{bmatrix},$$

$$N(z) = \begin{bmatrix} N_1(z) \\ N_2(z) \end{bmatrix}.$$

The asymptotic behavior of these functions in a neighborhood of the point at infinity can be described as

$$X^{-1}(z) \sim \begin{bmatrix} z^{-1} & 0 \\ z^{-2} & z^{-1} \end{bmatrix}, \quad z \rightarrow \infty,$$

$$N(z) = \begin{bmatrix} N_1^{(0)} \\ 0 \end{bmatrix} + o(1), \quad z \rightarrow \infty,$$

where

$$N_1^{(0)} = -\frac{3g}{2} \int_0^1 \eta^2 A_1(\eta) d\eta$$

according to (23).

Taking the behavior of $X^{-1}(z)$ and $N(z)$ at finite points of the complex plane and in the neighborhood of the point at infinity into account, we obtain a general solution of problem (26)

$$N(z) = -h_{as}(0, z) - A_0 \Phi(\eta_0, z) +$$

$$+ X(z) \left[\begin{bmatrix} C_1 \\ 0 \end{bmatrix} + \frac{1}{\eta_0 - z} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right],$$

or, in scalar form,

$$N_1(z) = -Bz - \frac{A_0 z}{\eta_0 - z} + U_1(z) \left[C_1 + \frac{d_1}{\eta_0 - z} \right], \quad (27)$$

$$N_2(z) = -\varepsilon_T + \frac{g_1}{g_2} \frac{A_0 \eta_0}{\eta_0 - z} +$$

$$+ \frac{g_1}{g_2} \left[U_0(z) - U_1(z) \right] \left[C_1 + \frac{d_1}{\eta_0 - z} \right] + \frac{d_2 U_0(z)}{\eta_0 - z}. \quad (28)$$

Here C_1 , d_1 and d_2 are constants that can be found from the solvability conditions for the boundary value problem.

We note that

$$U_0(z) = z - V_0^{-1} + o(1), \quad z \rightarrow \infty,$$

$$U_1(z) = z - V_1^{-1} + o(1), \quad z \rightarrow \infty,$$

where

$$V_1^{(-1)} = -\frac{1}{\pi} \int_0^1 \zeta_1(u) du = 0.84188,$$

$$V_0^{(-1)} = -\frac{1}{\pi} \int_0^1 \zeta_0(u) du = 0.71045,$$

Eliminating the pole of the function $N_1(z)$ given by equality (27) at the point at infinity, we obtain $C_1 = B$. Equating the limits on the right and on the left at the point $z = \infty$ in equality (27) for $N_1(z)$ and using (23), we obtain the equation

$$\frac{3g}{2} \int_0^1 \eta^2 A_1(\eta) d\eta = -A_0 + d_1 + B V_1^{(-1)}. \quad (29)$$

From the equation $N_2(\infty) = 0$, where the function $N_2(z)$ is given by equality (28), we find

$$\varepsilon_T = -d_2 + B \frac{g_1}{g_2} \left[V_1^{(-1)} - V_0^{(-1)} \right]. \quad (30)$$

Eliminating the poles of the functions $N_1(z)$ and $N_2(z)$ at the point η_0 , we obtain

$$d_1 = \frac{A_0 \eta_0}{U_1(\eta_0)}$$

and

$$d_2 = -\frac{g_1}{g_2}d_1 = -\frac{g_1}{g_2}\frac{A_0\eta_0}{U_1(\eta_0)}.$$

Taking these relations into account, we can rewrite equality (30) as

$$\varepsilon_T = \frac{g_1}{g_2} \left[\frac{A_0\eta_0}{U_1(\eta_0)} + B(V_1^{(-1)} - V_0^{(-1)}) \right]. \quad (31)$$

From definition (23) of the function $N(z)$, we have

$$N_1(z) = \frac{3g}{2}z \int_0^1 \frac{\eta^2 A_1(\eta) d\eta}{\eta - z},$$

whence

$$N_1^+(\mu) - N_1^-(\mu) = 3g\pi i \mu^3 A_1(\mu) \quad (32)$$

according to the Sokhotski formulas.

Substituting solution (27) in (32), we find

$$\begin{aligned} & \frac{3g}{2}z \eta^2 A_1(\eta) = \\ & + \frac{1}{2\pi i} \left[\left(C_1 + \frac{d_1}{\eta_0} \right) \frac{1}{\eta} - \frac{d_1}{\eta_0(\eta - \eta_0)} \right] [U_1^+(\eta) - U_1^-(\eta)]. \end{aligned} \quad (33)$$

Integrating equality (33) over η from 0 to 1, we obtain

$$\frac{3g}{2} \int_0^1 \eta^2 A_1(\eta) d\eta = \left(C_1 + \frac{d_1}{\eta_0} \right) J(0) - \frac{d_1}{\eta_0} J(\eta_0), \quad (34)$$

where

$$J(z) = \frac{1}{2\pi i} \int_0^1 \frac{U_1^+(\tau) - U_1^-(\tau)}{\tau - z} d\tau.$$

We give the integral representation of the function $U_1(z)$ without a derivation,

$$\begin{aligned} & U_1(z) - z + V_1^{(-1)} = \\ & = \frac{1}{2\pi i} \int_0^1 \frac{U_1^+(\tau) - U_1^-(\tau)}{\tau - z} d\tau, \quad z \notin [0, 1]. \end{aligned}$$

According to this representation, we have

$$J(\eta_0) = U_1(\eta_0) - \eta_0 + V_1^{(-1)}$$

and

$$J(0) = U_1(0) + V_1^{(-1)}.$$

Substituting these equalities in (34), we obtain the equation

$$\begin{aligned} & \frac{3g}{2} \int_0^1 \eta^2 A_1(\eta) d\eta = \left(C_1 + \frac{d_1}{\eta_0} \right) (U_1(0) + V_1^{(-1)}) - \\ & - \frac{d_1}{\eta_0} (U_1(\eta_0) - \eta_0 + V_1^{(-1)}). \end{aligned} \quad (35)$$

From Eqs. (29) and (35), we now obtain an equation from which we find

$$A_0 = -BU_1(\eta_0).$$

Substituting the found value A_0 in (31), we find that the temperature jump is given by

$$\varepsilon_T = B \frac{g_1}{g_2} \left[\eta_0 + V_1^{(-1)} - V_0^{(-1)} \right]. \quad (36)$$

7. TEMPERATURE JUMP

We express the temperature jump in terms of the heat flux, which is only transferred by the normal component and is proportional to its velocity [9]. We note that the mean velocity of the gas (the normal component and Bose condensate) is equal to zero in the gas volume. We calculate the heat flux using the formula

$$\mathbf{Q} = \int f \mathbf{v} \frac{mv^2}{2} \frac{(2s+1)d^3p}{(2\pi\hbar)^3}, \quad \mathbf{p} = m\mathbf{v}. \quad (37)$$

Passing to dimensionless quantities in the integral in (37), we obtain

$$\mathbf{Q} = \frac{(2s+1)m^4}{2(2\pi\hbar)^3\beta_s^6} \int [f_B^s + \varphi(x, \mathbf{C})g(C)] \mathbf{C} C^2 d^3C,$$

or

$$\begin{aligned} \mathbf{Q} &= \frac{(2s+1)mk^3T_s^3}{2\pi^3\hbar^3} \times \\ & \times \int [Ch_1(x, \mu) + C^2h_2(x, \mu)] \mathbf{C} C^2 g(C) d^3C. \end{aligned}$$

Taking into account that

$$\mathbf{C} = (C\mu, C \sin \theta \cos \chi, C \sin \theta \sin \chi),$$

$$d^3C = C^2 d\mu dC d\chi,$$

we obtain

$$Q_x(x) =$$

$$= Q_0 \int_{-1}^1 \int_0^\infty [Ch_1(x, \mu) + C^2 h_2(x, \mu)] \mu C^5 g(C) d\mu dC,$$

where

$$Q_0 = \frac{(2s+1)mk^3 T_s^3}{\pi^2 \hbar^3}.$$

We calculate the inner integral in (38),

$$Q_x = Q_0 \left(g_1 \int_{-1}^1 \mu h_1(x, \mu) d\mu + g_2 \int_{-1}^1 \mu h_2(x, \mu) d\mu \right).$$

Taking into account that the heat flux is conserved (i.e., $Q_x = \text{const}$), we find

$$Q_x = Q_0 g_1 \frac{2}{3} B,$$

hence

$$B = \frac{3Q_x}{2Q_0 g_1}. \quad (39)$$

Taking (39) into account, we represent (36) in the form

$$\Delta T = R Q_x, \quad (40)$$

where R is the Kapitza resistance,

$$R = \frac{3}{2g_2} \left[\eta_0 + V_1^{(-1)} - V_0^{(-1)} \right] \frac{\pi^2 \hbar^3}{(2s+1)mk^3 T_s^2},$$

or

$$\begin{aligned} R &= 0.58514 \frac{\pi^2 \hbar^3}{(2s+1)mk^3 T_s^2} = \\ &= 5.77510 \frac{\hbar^3}{(2s+1)mk^3 T_s^2}. \end{aligned} \quad (41)$$

Formula (40) is the sought temperature jump (the Kapitza jump) in the degenerate Bose gas. The coefficient R of the temperature jump, called the Kapitza resistance, is given by (41). It follows from relation (1) that obtained formula (41) is applicable under the condition

$$T \geq \frac{4\pi \hbar^2 a n}{mk}.$$

8. ANALYSIS OF RESULTS

It can be seen from (41) that the Kapitza resistance increases without bound as the temperature T_s

decreases. At present, there are no experimental data on the Kapitza resistance for a degenerate Bose gas. There exist only data on the Kapitza resistance for liquid helium ^4He [4], [9]. According to these data, there exists a divergence of the Kapitza resistance as $T_s \rightarrow 0$ in the case of liquid helium. The experimental data thus agree qualitatively with formula (41).

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